A GENERALIZATION OF CYCLIC AMENABILITY OF BANACH ALGEBRAS

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Abstract. This paper continues the investigation of Esslamzadeh and the first author which was begun in [7]. It is shown that homomorphic image of an approximately cyclic amenable Banach algebra is again approximately cyclic amenable. Equivalence of approximate cyclic amenability of a Banach algebra $\mathcal{A}$ and approximate cyclic amenability of $M_n(\mathcal{A})$ is proved. It is shown that under certain conditions the approximate cyclic amenability of second dual $\mathcal{A}^{**}$ implies the approximate cyclic amenability of $\mathcal{A}$.

1. Introduction

The concept of an amenable Banach algebra was defined and studied for the first time by Johnson in [12]. Since then, several modifications of this notion have been introduced by different authors (for instance, [1] and [9]) . The concept of cyclic amenability was presented by Gronbeak in [11]. He investigated the hereditary properties of this concept, found some relations between cyclic amenability of a Banach algebra and the trace extension property of its ideals. The notion of approximate amenability was introduced by Ghahramani and Loy [9] for Banach algebras were they characterized the structure of approximately amenable Banach algebras in several ways. They also gave some examples of approximately amenable, non-amenable Banach algebras to show that two notions of approximate amenability and Johnson's amenability do not coincide (for more information and examples refer also to [10]).

In this paper we define the concept of approximate cyclic amenability for Banach algebras and investigate the hereditary properties for this new notion. Furthermore, we show that for Banach algebras $\mathcal{A}$ and $\mathcal{B}$, if direct sum $\mathcal{A} \oplus \mathcal{B}$ with ℓ^1-norm is approximately cyclic amenable, then so are $\mathcal{A}$ and $\mathcal{B}$. By the means of an example we show that the converse is not true. However, the converse can be held if $\mathcal{A}^2$ is dense in $\mathcal{A}$. We also portray that approximate cyclic amenability of a Banach algebra $\mathcal{A}$ is equivalent to the approximate cyclic amenability of $M_n(\mathcal{A})$. In the third section, we show that homomorphic image

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of an approximately cyclic amenable Banach algebra under a continuous homomorphism is also approximately cyclic amenable. As a consequence, we prove that if the tensor product $\mathcal{A}\widehat{\otimes}\mathcal{B}$ is approximately cyclic amenable, then so are $\mathcal{A}$ and $\mathcal{B}$ provided that $\mathcal{A}$ and $\mathcal{B}$ admit nonzero character. Finally, some mild conditions can be imposed on $\mathcal{A}$ such that the approximate cyclic amenability of $\mathcal{A}^{**}$ with the first or the second Arens products, implies the approximate cyclic amenability of $\mathcal{A}$.

Let $\mathcal{A}$ be a Banach algebra and $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule. Then $\mathcal{X}^*$ is a Banach $\mathcal{A}$-bimodule with module actions

$$\langle a\cdot x^*, x\rangle = \langle x^*, x\cdot a\rangle \quad , \quad \langle x^*\cdot a, x\rangle = \langle x^*, a\cdot x\rangle \quad (a\in A, x\in X, x^*\in X^*).$$

A *derivation* from a Banach algebra $\mathcal{A}$ into a Banach $\mathcal{A}$-bimodule $\mathcal{X}$ is a bounded linear mapping $D:\mathcal{A}\longrightarrow\mathcal{X}$ such that $D(ab)=D(a)\cdot b+a\cdot D(b)$ for every $a,b\in A$. A derivation $D:\mathcal{A}\longrightarrow\mathcal{X}$ is called *inner* if there exists $x\in X$ such that $D(a)=a\cdot x-x\cdot a=\delta_x(a) \quad (a\in A)$. A Banach algebra $\mathcal{A}$ is called *amenable* if every bounded derivation $D:\mathcal{A}\longrightarrow\mathcal{X}^*$ is inner for every Banach $\mathcal{A}$-bimodule $\mathcal{X}$. A bounded derivation $D:\mathcal{A}\longrightarrow\mathcal{A}^*$ is called *cyclic* if $\langle Da,b\rangle+\langle Db,a\rangle=0$ for all $a,b\in\mathcal{A}$. A Banach algebra $\mathcal{A}$ is said to be *cyclic amenable* if every cyclic bounded derivation $D:\mathcal{A}\longrightarrow\mathcal{A}^*$ is inner. A derivation $D:\mathcal{A}\longrightarrow\mathcal{X}$ is called *approximately inner* if there exists a net $(x_\alpha)\subseteq\mathcal{X}$ such that

$$D(a)=\lim_\alpha(a\cdot x_\alpha-x_\alpha\cdot a)\quad(a\in\mathcal{A}).$$

Let $\mathcal{A}$ be an arbitrary Banach algebra. The first and second Arens multiplications on $\mathcal{A}^{**}$ which are denoted by "$\square$" and "$\Diamond$" respectively, are defined in three steps. For every $a,b\in\mathcal{A}, a^*\in\mathcal{A}^*$ and $a^{**},b^{**}\in A^{**}$, the elements $a^*\cdot a, a\cdot a^*, a^{**}\cdot a^*, a^*\cdot a^{**}$ of $\mathcal{A}^*$ and $a^{**}\square b^{**}, a^{**}\Diamond b^{**}$ of $\mathcal{A}^{**}$ are defined in the following way:

$$\langle a^*\cdot a,b\rangle=\langle a^*,ab\rangle,\quad \langle a\cdot a^*,b\rangle=\langle a^*,ba\rangle$$

$$\langle a^{**}\cdot a^*,a\rangle=\langle a^{**},a^*\cdot a\rangle,\quad \langle a^*\cdot a^{**},b\rangle=\langle a^{**},b\cdot a^*\rangle$$

$$\langle a^{**}\square b^{**},a^*\rangle=\langle a^{**},b^{**}\cdot a^*\rangle,\quad \langle a^{**}\Diamond b^{**},a^*\rangle=\langle b^{**},a^*\cdot a^{**}\rangle.$$

When these two products coincide on $\mathcal{A}^{**}$, we say that $\mathcal{A}$ is *Arens regular* (for more details refer to [2]).

2. Approximate cyclic amenability

We first recall the relevant material from [7], thus making our exposition self-contained.

Definition 2.1. A Banach algebra $\mathcal{A}$ is called approximately cyclic amenable if every cyclic derivation $D:\mathcal{A}\longrightarrow\mathcal{A}^*$ is approximately inner.

It is shown in [7, Example 4.3] that there is an approximately cyclic amenable Banach algebra which is not cyclic amenable. So the distinction between the cyclic amenability and the approximate cyclic amenability of Banach algebras are followed (see also Example 2.5).

Let $\mathcal{A}$ be a non-unital algebra. We denote by $\mathcal{A}^{\#}$, the unitization algebra of $\mathcal{A}$, formed by adjoining an identity to $\mathcal{A}$ so that $\mathcal{A}^{\#} = \mathcal{A} \oplus \mathbb{C}$, with the product

$$(a, \alpha)(b, \beta) = (ab + \beta a + \alpha b, \alpha\beta) \qquad (a, b \in \mathcal{A}, \quad \alpha, \beta \in \mathbb{C}).$$

In the case where $\mathcal{A}$ is a Banach algebra, $\mathcal{A}^{\#}$ is also a Banach algebra which contains $\mathcal{A}$ as a closed ideal. The following result shows the relationship of their approximate cyclic amenability which is proved in [7, Proposition 4.1].

Proposition 2.1. *Let $\mathcal{A}$ be a non-unital Banach algebra. The unitization algebra $\mathcal{A}^{\#}$ is approximately cyclic amenable if and only if $\mathcal{A}$ is approximately cyclic amenable.*

Let $\mathcal{I}$ be a closed ideal in $\mathcal{A}$. We say that $\mathcal{I}$ has *the approximate trace extension property* if for each $a^* \in \mathcal{I}^*$ with $a \cdot a^* = a^* \cdot a \, (a \in \mathcal{A})$ there is a net $(a^*_\alpha) \subseteq \mathcal{A}^*$ such that

$$a^*_\alpha \mid_{\mathcal{I}} = a^* (\text{for any } \alpha) \qquad \text{and} \qquad a \cdot a^*_\alpha - a^*_\alpha \cdot a \longrightarrow 0 \quad (a \in \mathcal{A}).$$

We also say that a bounded approximate identity $\{e_\alpha\}$ of $\mathcal{I}$ is *quasi central* for $\mathcal{A}$ if $\lim_\alpha \|ae_\alpha - e_\alpha a\| = 0$ for all $a \in \mathcal{A}$.

Proposition 2.2. *Let $\mathcal{A}$ be a Banach algebra with a closed ideal $\mathcal{I}$.*

(i) *Suppose that $\mathcal{A}/\mathcal{I}$ is approximately cyclic amenable. Then $\mathcal{I}$ has the approximate trace extension property;*
(ii) *Suppose that $\mathcal{A}$ is cyclic amenable and $\mathcal{I}$ has the approximate trace extension property. Then $\mathcal{A}/\mathcal{I}$ is approximately cyclic amenable;*
(iii) *If $\mathcal{A}/\mathcal{I}$ is approximately cyclic amenable, $\overline{\mathcal{I}^2} = \mathcal{I}$, and $\mathcal{I}$ is cyclic amenable, then $\mathcal{A}$ is approximately cyclic amenable;*
(iv) *Suppose that $\mathcal{A}$ is approximately cyclic amenable and $\mathcal{I}$ has a quasi-central bounded approximate identity for $\mathcal{A}$. Then $\mathcal{I}$ is approximately cyclic amenable.*

Proof. The proof of all parts is similar to [7, Proposition 2.2]. □

Example 4.4 of [7] shows that the condition $\overline{\mathcal{I}^2} = \mathcal{I}$ is necessary in Proposition 2.2 (iii). So this condition can not be removed.

Theorem 2.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras. Then the following statements hold:*

(i) *Suppose $\mathcal{A}$ is commutative and $\overline{\mathcal{A}^2} = \mathcal{A}$. If $\mathcal{A}$ and $\mathcal{B}$ are approximately cyclic amenable, then $\mathcal{A} \oplus \mathcal{B}$ is approximately cyclic amenable;*

(ii) *If $\mathcal{A} \oplus \mathcal{B}$ is approximately cyclic amenable, then both $\mathcal{A}$ and $\mathcal{B}$ are approximately cyclic amenable.*

Proof. (i): Since in the commutative case cyclic amenability and approximate cyclic amenability of Banach algebras are the same, the result follows from Proposition 2.2 (iii).

(ii): Without loss of generality and in view of Proposition 2.1, we see that $\mathcal{A}^{\#} \oplus \mathcal{B}^{\#}$ is approximately cyclic amenable. Moreover, $\mathcal{A}^{\#}$ and $\mathcal{B}^{\#}$ are ideals in $\mathcal{A}^{\#} \oplus \mathcal{B}^{\#}$ that have quasi central bounded approximate identities. Now by Proposition 2.2 (iv), $\mathcal{A}^{\#}$ and $\mathcal{B}^{\#}$ and as a result of Proposition 2.1, $\mathcal{A}$ and $\mathcal{B}$ are approximately cyclic amenable. □

In the general case, the converse of Theorem 2.3 (ii) is not true. Indeed, the condition $\overline{\mathcal{A}^2} = \mathcal{A}$ in part (i) is necessary as we will see in the following example.

Recall that a character on $\mathcal{A}$ is a non-zero homomorphism from $\mathcal{A}$ into $\mathbb{C}$. The set of characters on $\mathcal{A}$ is called the character space of $\mathcal{A}$ and is denoted by $\Phi_{\mathcal{A}}$. Let $\varphi \in \Phi_{\mathcal{A}} \cup \{0\}$. Then a linear functional $d \in \mathcal{A}^*$ is a point derivation at φ if

$$d(ab) = \varphi(a)d(b) + \varphi(b)d(a) \quad (a, b \in \mathcal{A}).$$

Example 1. (i) Let $\mathcal{A}$ be a non-zero Banach algebra with zero product, that is $ab = 0$ for all $a, b \in \mathcal{A}$. It is shown in [11, Example 2.5] that such Banach algebra is cyclicly amenable if and only if its dimension is one. Consider $\mathcal{A} = \mathbb{C}$ with the zero product. Then $\mathcal{A}$ is approximately cyclic amenable. Since the properties of cyclic amenability and approximate cyclic amenability are the same for commutative Banach algebras, $\mathcal{A} \oplus \mathcal{A}$ is not approximately cyclic amenable.

(ii) Let $\mathcal{A}$ be a [approximately] weakly amenable Banach algebra. Then $\overline{\mathcal{A}^2} = \mathcal{A}$. But from part (i) we conclude that in the general case this condition is not necessary when $\mathcal{A}$ is [approximately] cyclic amenable. Meanwhile, we have seen that $\mathcal{A}$ is [approximately] cyclicly amenable, while it is not [approximately] weakly amenable.

(iii) It is known that if $\mathcal{A}$ [approximately] weakly amenable then there is no non-zero continuous point derivation on $\mathcal{A}$. This is not true for [approximate] cyclic amenability. In other words, if $\mathcal{A} = \mathbb{C}$ with the zero product, then for every non-zero map $d \in \mathcal{A}^*$ and zero map φ, d is a non-zero continuous point derivation at φ. However $\mathcal{A}$ is [approximately] cyclic amenable.

Let $\mathcal{A}$ be a Banach algebra, $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule and $n \in \mathbb{N}$. We shall regard $M_n(\mathcal{X})$ as a Banach $M_n(\mathcal{A})$-bimodule in the obvious way so that

$$(a \cdot x)_{ij} = \sum_{k=1}^{n} a_{ik} \cdot x_{kj} \qquad (a = (a_{ij}) \in M_n(\mathcal{A}), x = (x_{ij}) \in M_n(\mathcal{X})).$$

We have the identity

$$M_n(\mathcal{X})^* = M_n(\mathcal{X}^*) = \mathcal{X}^* \widehat{\otimes} M_n$$

with duality

$$\langle \Lambda, x\rangle = \sum_{i,j=1}^{n} \langle \lambda_{ij}, x_{ij}\rangle, \qquad (x = (x_{ij}) \in M_n(\mathcal{X}), \Lambda = (\lambda_{ij}) \in M_n(\mathcal{X}^*)).$$

Note that

$$(a \cdot \Lambda)_{ij} = \sum_{k=1}^{n} a_{jk} \cdot \lambda_{ik} \quad \text{and} \quad (\Lambda \cdot a)_{ij} = \sum_{k=1}^{n} \lambda_{kj} \cdot a_{ki}. \tag{2.1}$$

for $a = (a_{ij}) \in M_n(\mathcal{A})$ and $\Lambda = (\lambda_{ij}) \in M_n(\mathcal{X})^*$. One should remember that $M_n(\mathcal{A})$ is isometrically algebra isomorphic to $M_n \widehat{\otimes} \mathcal{A}$.

Theorem 2.4. *Let $\mathcal{A}$ be a Banach algebra and $n \in \mathbb{N}$. Then $\mathcal{A}$ is approximately cyclic amenable if and only if $M_n(\mathcal{A})$ is approximately cyclic amenable.*

Proof. If $\mathcal{A}$ does not have an identity, according to equality $M_n(\mathcal{A})^\# = M_n(\mathcal{A}^\#)$ and Proposition 2.1, we can assume that $\mathcal{A}$ has an identity. Let $\mathcal{A}$ be an approximately cyclic amenable Banach algebra and $D : M_n(\mathcal{A}) \longrightarrow M_n(\mathcal{A})^*$ be a cyclic continuous derivation. We regard M_n as a subalgebra of $M_n(\mathcal{A})$. Since M_n is amenable, there exists an element $\Lambda = (\lambda_{ij}) \in M_n(\mathcal{A}^*)$ with $D|_{M_n} = ad_\Lambda$. Replacing D by $D - ad_\Lambda$, we may suppose $D|_{M_n} = 0$. For $r, s \in \mathbb{N}_n$ and $a \in \mathcal{A}$, let

$$D((a)_{rs}) = (d_{ij}^{(r,s)} : i, j \in \mathbb{N}_n) \in M_n(\mathcal{A}^*).$$

We have

$$D((a)_{rs}) = D(\varepsilon_{r1}(a)_{11}\varepsilon_{1s}) = \varepsilon_{r1} \cdot D((a)_{11}) \cdot \varepsilon_{1s}.$$

Since $D(\varepsilon_{r1}) = D(\varepsilon_{1s}) = 0$, by (2.1) we have $d_{ij}^{(r,s)} = 0$ $(i, j \in \mathbb{N}_n)$ except when $(i, j) = (r, s)$ and in this case $d_{sr}^{(r,s)} = d_{11}^{(1,1)}$. Putting $d_{11}^{(1,1)} = d(a)$, the map

$$d : \mathcal{A} \longrightarrow \mathcal{A}^*; a \mapsto d(a)$$

is a cyclic continuous derivation because for $a, b \in \mathcal{A}$, by (2.1) we get

$$D((ab)_{rs}) = D((a)_{r1} \cdot (b)_{1s}) = D((a)_{r1}) \cdot (b)_{1s} + (a)_{r1} \cdot D((b)_{1s}).$$

This implies that $d(ab) = d(a) \cdot b + a \cdot d(b)$. On the other hand,

$$\langle d(a), b\rangle + \langle d(b), a\rangle = \langle D((a)_{sr}), (b)_{rs}\rangle + \langle D((b)_{rs}), (a)_{sr}\rangle = 0,$$

for all $a, b \in \mathcal{A}$. Due to approximate cyclic amenability of $\mathcal{A}$, there exists a net $(\lambda_\alpha) \subseteq \mathcal{A}^*$ such that

$$d(a) = \lim_\alpha a \cdot \lambda_\alpha - \lambda_\alpha \cdot a \quad (a \in \mathcal{A}).$$

Take $\Lambda_\alpha \in M_n(\mathcal{A}^*)$ to be the matrix that has λ_α in each diagonal position and zero elsewhere. Then by (2.1) we see

$$D((a_{ij})) = \lim_\alpha (a_{ij}) \cdot \Lambda_\alpha - \Lambda_\alpha \cdot (a_{ij}) \qquad ((a_{ij}) \in M_n(\mathcal{A}^*)).$$

This shows that $M_n(\mathcal{A})$ is approximately cyclic amenable.

Conversely, suppose that $M_n(\mathcal{A})$ is approximately cyclic amenable and $D : \mathcal{A} \longrightarrow \mathcal{A}^*$ is a continuous cyclic derivation. It is easy to check that $D \otimes 1 : \mathcal{A}\widehat{\otimes}M_n \longrightarrow \mathcal{A}^*\widehat{\otimes}M_n = \mathcal{A}^* \otimes M_n$ is a continuous cyclic derivation. Therefore there exists a net $(a_\alpha) \subseteq \mathcal{A}^* \otimes M_n$ such that $a_\alpha = \sum_{i,j=1}^n a_{ij}^\alpha \otimes \varepsilon_{ij}$ and

$$D \otimes 1(\mathcal{B}) = \lim_\alpha (\mathcal{B} \cdot a_\alpha - a_\alpha \cdot \mathcal{B}) \qquad \mathcal{B} \in \mathcal{A}\widehat{\otimes}M_n.$$

Thus for every $a \in \mathcal{A}$, we have

$$\begin{aligned} D(a) \otimes \varepsilon_{11} &= (D \otimes 1)(a \otimes \varepsilon_{11}) \\ &= \lim_\alpha ((a \otimes \varepsilon_{11}) \cdot a_\alpha - a_\alpha \cdot (a \otimes \varepsilon_{11})) \\ &= \lim_\alpha (\sum_{i=1}^n a a_{i1}^\alpha \otimes \varepsilon_{i1} - \sum_{j=1}^n a_{1j}^\alpha a \otimes \varepsilon_{1j}). \end{aligned}$$

Hence

$$\begin{pmatrix} Da & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} = \lim_\alpha (\begin{pmatrix} aa_{11}^\alpha & 0 & \dots & 0 \\ aa_{21}^\alpha & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ aa_{n1}^\alpha & 0 & \dots & 0 \end{pmatrix} - \begin{pmatrix} a_{11}^\alpha a & a_{12}^\alpha a & \dots & a_{1n}^\alpha a \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}),$$

The above equality implies that $D(a) = \lim_\alpha aa_{11}^\alpha - a_{11}^\alpha a$. Therefore $\mathcal{A}$ is approximately cyclic amenable. $\square$

3. Approximate cyclic amenability of second dual

For a Banach algebra $\mathcal{A}$ let $\mathcal{A}^{op}$ be a Banach algebra, whose underlying Banach space is $\mathcal{A}$, but the product is $\circ$, where $a \circ b = ba$ in which $a, b \in \mathcal{A}$.

Lemma 3.1. *Let $\mathcal{A}$ be a Banach algebra. Then $\mathcal{A}$ is approximately cyclic amenable if and only if $\mathcal{A}^{op}$ is approximately cyclic amenable.*

Proof. The identity map $i : \mathcal{A} \longrightarrow \mathcal{A}^{op}$ is a continuous anti-isomorphism. Suppose that the map $D : \mathcal{A}^{op} \longrightarrow (A^{op})^*$ is a continuous cyclic derivation. It is easy to see that $i^* \circ D \circ i : \mathcal{A} \longrightarrow \mathcal{A}^*$ is a cyclic derivation. Since $\mathcal{A}$ is approximately cyclic amenable, there exists a net $(x_\alpha) \subseteq \mathcal{A}^*$ such that

$$i^* \circ D \circ i(a) = \lim_\alpha (a \cdot x_\alpha - x_\alpha \cdot a) \quad (a \in \mathcal{A}).$$

Moreover $(i^*)^2 = I_{(\mathcal{A}^{op})^*}$. Applying i^* to both sides of the above equation, we have

$$D(a) = D(i(a)) = \lim_{\alpha}(a \cdot x_\alpha - x_\alpha \cdot a) \quad (a \in \mathcal{A}).$$

This implies that $\mathcal{A}^{op}$ is approximately cyclic amenable. Since $(\mathcal{A}^{op})^{op} = \mathcal{A}$, the proof of converse is done similarly. $\square$

We need the following result which has been proven in [7, Proposition 4.6].

Proposition 3.1. *Let $\mathcal{A}$ be a Banach algebra, $\mathcal{B}$ be a closed subalgebra of $\mathcal{A}$ and $\mathcal{I}$ be a closed ideal are in $\mathcal{A}$ such that $\mathcal{A} = \mathcal{B} \oplus \mathcal{I}$. If $\mathcal{A}$ is approximately cyclic amenable, then so is $\mathcal{B}$.*

Recall that a Banach algebra $\mathcal{A}$ is said to be a dual Banach algebra if there is a closed submodule $\mathcal{A}_*$ of $\mathcal{A}^*$ such that $\mathcal{A} = (\mathcal{A}_*)^*$.

Theorem 3.2. *Let $\mathcal{A}$ be a dual Banach algebra. If $(\mathcal{A}^{**}, \square)$ is approximately cyclic amenable then $\mathcal{A}$ is approximately cyclic amenable.*

Proof. According to [3, Theorem 2.15], $(\mathcal{A}_*)^\perp$ is a ω^*-closed ideal in $\mathcal{A}^{**}$ and $\mathcal{A}^{**} = \mathcal{A} \oplus (\mathcal{A}_*)^\perp$. Since $\mathcal{A}^{**}$ is approximately cyclic amenable, $\mathcal{A}$ is approximately cyclic amenable by Proposition 3.1. $\square$

We have shown in Propositions 2.2 (ii) and 3.1 that under certain conditions, the homomorphic image of an approximately cyclic amenable Banach algebra is again an approximately cyclic amenable. In the upcoming theorem, we extend Proposition 3.1 by using homomorphisms.

Theorem 3.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras. If $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ and $\psi : \mathcal{B} \longrightarrow \mathcal{A}$ are continuous homomorphisms such that $\varphi \circ \psi = I_{\mathcal{B}}$, then*

(i) *If $\mathcal{A}$ is approximately cyclic amenable, then so is $\mathcal{B}$;*
(ii) *If $(\mathcal{A}^{**}, \square)$ is approximately cyclic amenable, then so is $(\mathcal{B}^{**}, \square)$.*

Proof. (i) Let $D : \mathcal{B} \longrightarrow \mathcal{B}^*$ be a cyclic derivation. The map φ^* is an $\mathcal{A}$-module homomorphism, and so

$$\begin{aligned} \varphi^* \circ D \circ \varphi(ab) &= \varphi^*(D(\varphi(a)) \cdot \varphi(b) + \varphi(a) \cdot D(\varphi(b))) \\ &= \varphi^* \circ D \circ \varphi(a) \cdot b + a \cdot \varphi^* \circ D \circ \varphi(b), \end{aligned}$$

for all $a, b \in \mathcal{A}$. Hence, $\varphi^* \circ D \circ \varphi : \mathcal{A} \longrightarrow \mathcal{A}^*$ is a continuous derivation. Moreover, $\varphi^* \circ D \circ \varphi$ is cyclic, since D is cyclic. Therefore there exists a net $(a^*_\alpha) \subseteq \mathcal{A}^*$ such that $\varphi^* \circ D \circ \varphi(a) = \lim_\alpha (a \cdot a^*_\alpha - a^*_\alpha \cdot a) \quad (a \in \mathcal{A})$. The equality $\varphi \circ \psi = I_{\mathcal{B}}$ implies $\psi^* \circ \varphi^* = I_{\mathcal{B}^*}$. For every $c \in \mathcal{B}$, we get

$$\begin{aligned}D(c) &= \psi^* \circ \varphi^* \circ D \circ \varphi \circ \psi(c)\\ &= \psi^*(\varphi^* \circ D \circ \varphi(\psi(c))\\ &= \psi^*(\lim_\alpha(\psi(c) \cdot a_\alpha^* - a_\alpha^* \cdot \psi(c)))\\ &= \lim_\alpha \psi^*(\psi(c) \cdot a_\alpha^* - a_\alpha^* \cdot \psi(c))\\ &= \lim_\alpha(c \cdot \psi^*(a_\alpha^*) - \psi^*(a_\alpha^*) \cdot c).\end{aligned}$$

It follows that $\mathcal{B}$ is approximately cyclic amenable.

(ii) Since $\varphi^{**} \circ \psi^{**} = I_{\mathcal{B}^{**}}$, the proof is similar to (i). □

Theorem 3.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras. Assume that the charater spaces of $\mathcal{A}$ and $\mathcal{B}$ are non empty.*

(i) *If $\mathcal{A}\widehat{\otimes}\mathcal{B}$ is approximately cyclic amenable, then so are $\mathcal{A}$ and $\mathcal{B}$;*
(ii) *If $\mathcal{A}$ is commutative and $\mathcal{A}\widehat{\otimes}\mathcal{A}$ is approximately cyclic amenable, then $\mathcal{A}$ is approximately cyclic amenable.*

Proof. By Proposition 2.1 we may suppose that $\mathcal{A}$ and $\mathcal{B}$ are unital with identities $e_{\mathcal{A}}$ and $e_{\mathcal{B}}$.

(i) It is obvious that the map $\varphi : \mathcal{A}\widehat{\otimes}\mathcal{B} \longrightarrow \mathcal{A}$: $a \otimes b \longmapsto \phi(b)a$ is a continuous linear map, where $\phi \in \Phi_{\mathcal{B}}$. Take $u, v \in \mathcal{A}\widehat{\otimes}\mathcal{B}$ such that $u = \sum_{n=1}^{\infty} a_n \otimes b_n$ and $v = \sum_{n=1}^{\infty} c_m \otimes d_m \quad (a_n, c_m \in \mathcal{A}, b_n, d_m \in \mathcal{B})$. Hence,

$$\begin{aligned}\varphi(uv) &= \varphi(\sum_{n,m=1}^{\infty} a_n c_m \otimes b_n d_m)\\ &= \sum_{n,m=1}^{\infty} \phi(b_n d_m) a_n c_m = \sum_{n,m=1}^{\infty} \phi(b_n)\phi(d_m) a_n c_m\\ &= (\sum_{n=1}^{\infty} \phi(b_n) a_n)(\sum_{m=1}^{\infty} \phi(d_m) c_m) = \varphi(u)\varphi(v).\end{aligned}$$

So, φ is a continuous homomorphism. Moreover, the map $\psi : \mathcal{A} \longrightarrow \mathcal{A}\widehat{\otimes}\mathcal{B}$ $a \longmapsto a \otimes e_{\mathcal{B}}$ is continuous homomorphism such that $\varphi \circ \psi = I_{\mathcal{A}}$. As a result of Theorem 3.3 (i), $\mathcal{A}$ is approximately cyclic amenable and likewise $\mathcal{B}$ is approximately cyclic amenable.

(ii) Consider the homomorphisms $\varphi : \mathcal{A}\widehat{\otimes}\mathcal{A} \longrightarrow \mathcal{A}$: $a\widehat{\otimes}b \longmapsto ab$ and $\psi : \mathcal{A} \longrightarrow \mathcal{A}\widehat{\otimes}\mathcal{A}$: $a \longmapsto a \otimes e_{\mathcal{A}}$. Thus $\varphi \circ \psi = I_{\mathcal{A}}$. Now, Theorem 3.3 (i) shows that $\mathcal{A}$ is approximately cyclic amenable. □

In the rest of the paper, we investigate conditions under which approximate cyclic amenability of $\mathcal{A}^{**}$ necessitates approximate cyclic amenability of $\mathcal{A}$.

Let $\mathcal{A}$ be a Banach algebra. The space of almost periodic functionals on $\mathcal{A}$ is defined by

$$WAP(A) = \{a^* \in \mathcal{A}^* : a \longmapsto a \cdot a^*;\ \mathcal{A} \longrightarrow \mathcal{A}^* \text{ is weakly compact}\}.$$

Also, the topological center $Z(\mathcal{A}^{**})$ of $\mathcal{A}^{**}$ is defined by

$$Z(\mathcal{A}^{**}) = \{b^{**} : \text{The map } a^{**} \mapsto b^{**}\Box a^{**} \text{ is } \omega^* - \omega^*\text{-continuous}\}.$$

Theorem 3.5. *Let $\mathcal{A}$ be a Banach algebra.*

(i) *Suppose $(\mathcal{A}^{**}, \Box)$ is approximately cyclic amenable and every cyclic derivation $D : \mathcal{A} \longrightarrow \mathcal{A}^*$ satisfies $D^{**}(\mathcal{A}^{**}) \subseteq WAP(\mathcal{A})$. Then $\mathcal{A}$ is approximately cyclic amenable;*

(ii) *Suppose $\mathcal{A}$ is Arens regular, every derivation $D : \mathcal{A} \longrightarrow \mathcal{A}^*$ is weakly compact and $\mathcal{A}^{**}$ is approximately cyclic amenable. Then $\mathcal{A}$ is approximately cyclic amenable.*

Proof. (i) Let $D : \mathcal{A} \longrightarrow \mathcal{A}^*$ be a cyclic derivation. It follows from [6, Theorem 2.1] that $D^{**} : \mathcal{A}^{**} \longrightarrow \mathcal{A}^{***}$ is a derivation. For every $a^{**}, b^{**} \in \mathcal{A}^{**}$ take two bounded nets $(a_\alpha), (b_\beta) \subseteq \mathcal{A}$ such that $a^{**} = \omega^* - \lim_\alpha a_\alpha$ and $b^{**} = \omega^* - \lim_\beta b_\beta$. Hence, $\langle D^{**}(a^{**}), b^{**}\rangle = \lim_\alpha \lim_\beta \langle D(a_\alpha), b_\beta\rangle$ and since $D^{**}(a^{**}) \subseteq WAP(\mathcal{A}) \subseteq \mathcal{A}^*$, $\langle D^{**}(b^{**}), a^{**}\rangle = \lim_\alpha \lim_\beta \langle D(b_\beta), a_\alpha\rangle$. The above equalities show that

$$\langle D^{**}(a^{**}), b^{**}\rangle + \langle D^{**}(b^{**}), a^{**}\rangle = \lim_\alpha \lim_\beta (\langle D(a_\alpha), b_\beta\rangle + \langle Db_\beta, a_\alpha\rangle) = 0.$$

Thus D^{**} is cyclic, and so there exists a net $(a_\alpha^{***}) \subseteq \mathcal{A}^{***}$ such that

$$D^{**}(a^{**}) = \lim_\alpha (a^{**} \cdot a_\alpha^{***} - a_\alpha^{***} \cdot a^{**}) \quad (a^{**} \in \mathcal{A}^{**}).$$

On the other hand,

$$D(a) = P(D(a)) = \lim_\alpha (a \cdot P(a_\alpha^{***}) - P(a_\alpha^{***}) \cdot a) \quad (a \in \mathcal{A}),$$

where $P : \mathcal{A}^{***} \longrightarrow \mathcal{A}^*$ is the natural projection. Therefore D is approximately inner and thus $\mathcal{A}$ is approximately cyclic amenable.

(ii) Suppose $D : \mathcal{A} \longrightarrow \mathcal{A}^*$ is a cyclic derivation. Since D is weakly compact, $D^{**}(\mathcal{A}^{**}) \subseteq \mathcal{A}^*$ and by [5, Corollary 7.2(i)], $D^{**} : \mathcal{A}^{**} \longrightarrow \mathcal{A}^{***}$ is a derivation. So similar to part (i), D^{**} is cyclic and approximately inner. Therefore, we obtain the desired result. □

Theorem 3.6. *Let $\mathcal{A}$ be a Banach algebra and every cyclic derivation $D : \mathcal{A} \longrightarrow \mathcal{A}^*$ is weakly compact. If the topological center $Z(\mathcal{A}^{**})$ is approximately cyclic amenable, then so is $\mathcal{A}$.*

Proof. For simplicity, we put $\mathcal{B} = Z(\mathcal{A}^{**})$. Let $D : \mathcal{A} \longrightarrow \mathcal{A}^*$ be a cyclic derivation and $J : \mathcal{B} \longrightarrow \mathcal{A}^{**}$ is the inclusion map. Let us consider the map

$\widetilde{D} := J^* \circ D^{**} \circ J : \mathcal{B} \longrightarrow \mathcal{B}^*$. Assume that $a^{**}, b^{**} \in \mathcal{B}$ and choose two bounded nets $(a_\alpha), (b_\beta) \subseteq \mathcal{A}$ such that $a^{**} = \omega^* - \lim_\alpha a_\alpha$ and $b^{**} = \omega^* - \lim_\beta b_\beta$. Then

$$\begin{aligned}
\widetilde{D}(a^{**}b^{**}) &= \lim_\alpha \lim_\beta \widetilde{D}(a_\alpha b_\beta) \\
&= \lim_\alpha \lim_\beta J^*(D(a_\alpha) \cdot b_\beta + a_\alpha \cdot D(b_\beta)) \\
&= \lim_\alpha \lim_\beta (J^* \circ D(a_\alpha) \cdot b_\beta + a_\alpha \cdot J^* \circ D(b_\beta)).
\end{aligned}$$

Note that in the last equality we used the fact that J^* is a $\mathcal{B}$-module homomorphism. Obviously, $\lim_\alpha \lim_\beta J^* \circ D(a_\alpha) \cdot b_\beta = \widetilde{D}(a^{**}) \cdot b^{**}$. Since the map $a^{**} \mapsto b^{**}a^{**}$, $\omega^* - \omega^*$-continuous, $\lim_\alpha \lim_\beta a_\alpha \cdot J^* \circ D(b_\beta) = a^{**} \cdot \widetilde{D}(b^{**})$ and as a result $\widetilde{D}$ is a derivation. Moreover, $\langle \widetilde{D}(a^{**}), b^{**} \rangle = \lim_\alpha \lim_\beta \langle D(a_\alpha), b_\beta \rangle$. By assumption since D is weakly compact we have $D^{**}(\mathcal{A}^{**}) \subseteq \mathcal{A}^*$ and so $\langle \widetilde{D}(b^{**}), a^{**} \rangle = \lim_\alpha \lim_\beta \langle D(b_\beta), a_\alpha \rangle$. Hence

$$\begin{aligned}
0 &= \lim_\alpha \lim_\beta (\langle D(a_\alpha), b_\beta \rangle + \langle D(b_\beta), a_\alpha \rangle) \\
&= \langle \widetilde{D}(a^{**}), b^{**} \rangle + \langle \widetilde{D}(b^{**}), a^{**} \rangle.
\end{aligned}$$

Therefore, $\widetilde{D}$ is a cyclic derivation and since $\mathcal{B}$ is approximately cyclic amenable then there exists a net $(b^*_\gamma) \subseteq \mathcal{B}^*$ such that

$$\widetilde{D}(b) = \lim_\gamma (b \cdot b^*_\gamma - b^*_\gamma \cdot b) \quad (b \in \mathcal{B}).$$

Now, set $a^*_\gamma = b^*_\gamma|_{\mathcal{A}}$. By considering the net $(a^*_\gamma) \subseteq \mathcal{A}^*$ and above relation we have

$$D(a) = \lim_\gamma (a \cdot a^*_\gamma - a^*_\gamma \cdot a) \quad (a \in \mathcal{A}).$$

Therefore $\mathcal{A}$ is approximately cyclic amenable. $\square$

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